Compact endomorphisms of $H^\infty(D)$

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February 1, 2008

Let D be the open unit disc and, as usual, let $H^\infty(D)$ be the algebra of bounded analytic functions on D with $\|f\| = \sup_{z\in D}|f(z)|$. With pointwise addition and multiplication, $H^\infty(D)$ is a well known uniform algebra. In this note we characterize the compact endomorphisms of $H^\infty(D)$ and determine their spectra.

We show that although not every endomorphism T of $H^\infty(D)$ has the form $T(f)(z) = f(\phi(z))$ for some analytic ϕ mapping D into itself, if T is compact, there is an analytic function $\psi : D \to D$ associated with T. In the case where T is compact, the derivative of ψ at its fixed point determines the spectrum of T.

The structure of the maximal ideal space M_{H^∞} is well known. Evaluation at a point $z \in D$ gives rise to an element in M_{H^∞} in the natural way. The remainder of M_{H^∞} consists of singleton Gleason parts and Gleason parts which are analytic discs. An analytic disc, $P(m)$, containing a point $m \in M_{H^\infty}$, is a subset of M_{H^∞} for which there exists a continuous bijection $L_m : D \to P(m)$ such that $L_m(0) = m$ and $\hat{f}(L_m(z))$ is analytic on D for each $f \in H^\infty(D)$. Moreover, the map L_m has the form $L_m(z) = w^* \lim \dfrac{z+z_\alpha}{1+\overline{z_\alpha}z}$ for some net $z_\alpha \to m$ in the w*topology, whence $\hat{f}(L_m(z)) = \lim f(\dfrac{z+z_\alpha}{1+\overline{z_\alpha}z})$ for all $f \in H^\infty(D)$. A fiber M_λ over some $\lambda \in \overline{D}\setminus D$, is the zero set in M_{H^∞} of the function $z-\lambda$. Each part, distinct from D, is contained in exactly one fiber M_λ. With no loss of generality we let $\lambda = 1$. We recall, too, that two elements n_1 and n_2 are in the same part if, and only if, $\|n_1 - n_2\| < 2$, where $\|.\|$ is the norm in the dual space $H^\infty(D)^*$.

Now let T be an endomorphism of $H^\infty(D)$, i.e. T is a (necessarily) bounded linear map of $H^\infty(D)$ to itself with $T(fg) = T(f)T(g)$ for all $f, g \in$

$H^\infty(D)$. For a given T, either T has the form $Tf(z) = f(\omega(z))$ for some analytic map $\omega : D \to D$, or $Tf = \hat{f}(n)1$ for some $n \in M_{H^\infty}$, or there exists an $m \in M_{H^\infty}$, a net $z_\alpha \to m$ in the w* topology and an analytic function $\tau : D \to D$, with $\tau(0) = 0$ for which $Tf(z) = \hat{f}(L_m(\tau(z)))$ [3]. Further, on general principles, if T is an endomorphism of $H^\infty(D)$ there exists a w* continuous map $\phi : M_{H^\infty} \to M_{H^\infty}$ with $\widehat{Tf}(n) = \hat{f}(\phi(n))$ for all $n \in M_{H^\infty}$. In the last case, $\phi(z) = L_m(\tau(z))$ for $z \in D$.

For a given endomorphism T, if the induced map ϕ maps D to itself, then T is commonly called a *composition operator*. Compact composition operators on H^∞ were completely characterized in [4]. However, in general, $L_m(\tau(z))$ need not be in D, and so not every endomorphism of $H^\infty(D)$ is a composition operator. It is these endomorphisms that we discuss here. Trivially, for any $n \in M_{H^\infty} \setminus D$, the map T defined by $T : Tf(z) = \hat{f}(n)1$ is a compact endomorphism of $H^\infty(D)$ which is not a composition operator.

Now let $P(m)$ be an analytic part and let T be an endomorphism defined by $Tf(z) = \hat{f}(L_m(\tau(z)))$ as discussed above. Also suppose that $\phi : M_{H^\infty} \to M_{H^\infty}$ is such that $\widehat{Tf} = \hat{f} \circ \phi$. Assume that T is compact. We claim that $\overline{\tau(D)}$ is a compact subset of D in the Euclidean topology. Indeed, if we regard the endomorphism T as an operator from $H^\infty(D)$ into $C(M_{H^\infty})$, then T is compact if, and only if, ϕ is w* to norm continuous on M_{H^∞} [2]. Since M_{H^∞} is itself compact and connected (in the w* topology), $\phi(M_{H^\infty})$ must be compact and connected in the norm topology on M_{H^∞} and so ϕ maps M_{H^∞} into a norm compact connected subset of $P(m)$. Therefore the range, $\phi(D) = L_m(\tau(D))$ is contained in a norm compact subset of $P(m)$, and further since L_m^{-1} is an isometry in the Gleason norms on $P(m)$ and D [1], $\tau(D) = L_m^{-1}(\phi(D))$ is contained in a compact subset of D in the norm topology on D. Since the norm, Euclidean and w* topologies on D coincide, $\overline{\tau(D)}$ is a compact subset of D in these three topologies. As a consequence, $\hat{\tau}(M_{H^\infty}) \subset D$.

Next consider two maps of $H^\infty(D)$ into itself. The first, C_{L_m} is defined by $C_{L_m}(f)(z) = \hat{f}(L_m(z))$, and the second C_τ by $C_\tau(f)(z) = f(\tau(z))$. Then $(C_{L_m} \circ C_\tau)(f)(z) = C_{L_m}(f \circ \tau)(z) = \widehat{f \circ \tau}(L_m(z))$ and $(C_\tau \circ C_{L_m})(f)(z) = \hat{f}(L_m(\tau(z))) = Tf(z)$. But if B is a Banach space and S_1 and S_2 are any two bounded linear maps from $B \to B$, the spectrum $\sigma(S_1S_2) \setminus \{0\} = \sigma(S_2S_1) \setminus \{0\}$. Thus we see that $\sigma(T) \setminus \{0\} = \sigma(C_{L_m} \circ C_\tau) \setminus \{0\}$.

Since f is analytic on a neighborhood of range $\hat{\tau}$ which is a subset of D, a standard functional calculus argument gives $\widehat{f \circ \tau}(L_m(z)) = f(\hat{\tau}(L_m(z))$.

If we let $\psi(z) = \hat{\tau}(L_m(z))$ we see that $C_{L_m} \circ C_\tau$ is a compact composition operator in the usual sense, and so if $z_0 \in D$ is the unique fixed point of ψ, and $\mathbf{N}$ is the set of positive integers, then $\sigma(T) = \{(\psi'(z_0))^n : n \in \mathbf{N}\} \cup \{0, 1\}$.

To summarize, we have shown the following.

Theorem: If T is a compact endomorphism of $H^\infty(D)$, then either T has one dimensional range, i.e. $Tf = \hat{f}(n)1$ for some $n \in M_{H^\infty}$, or T is a composition operator in the usual sense, or T has the form $Tf(z) = \hat{f}(L_m(\tau(z)))$ where τ is described above. In the last case, there is a compact composition operator C_ψ, such that $\sigma(T) = \sigma(C_\psi) = \{(\psi'(z_0))^n : n \in \mathbf{N}\} \cup \{0, 1\}$ where $z_0 \in D$ is the unique fixed point of ψ.

We conclude with two examples showing differences between composition operators and general endomorphisms .

(a) With the same terminology and symbols, suppose $\hat{\tau}$ is constant on $P(m)$, i.e. $\hat{\tau}(P(m)) = \{\hat{\tau}(m)\}$. Since T is compact, $\hat{\tau}(m) \in D$. Then using C_τ and C_{L_m} as before, we show that $T^2 f = \hat{f}(n)1$ for some $n \in P(m)$. Indeed, $(C_{L_m} \circ C_\tau)f = f(t_0)1$ where $t_0 = \hat{\tau}(m) \in D$. Then we see that

$$T^2 f = [(C_\tau \circ C_{L_m}) \circ (C_\tau \circ C_{L_m})]f =$$

$$[C_\tau \circ (C_{L_m} \circ C_\tau) \circ C_{L_m}]f = [C_\tau \circ (C_{L_m} \circ C_\tau)](\hat{f} \circ L_m) = C_\tau(\hat{f}(L_m(t_0))1) = \hat{f}(L_m(t_0))1.$$

Letting $n = L_m(t_0)$ gives the result.

One way to have $\hat{\tau}$ constant on $P(m)$ is for τ to be continuous at 1 in the usual sense.

A more interesting example, perhaps, is to define τ by $\tau(z) = \frac{1}{2} z e^{\frac{z+1}{z-1}}$, and $m \in M_{H^\infty}$ as a w* limit of a real net x_α approaching 1. Then $\hat{\tau}(L_m(z)) = \lim_\alpha \tau(\frac{z + x_\alpha}{1 + \overline{x_\alpha} z}) = 0$, and so $T^2 f = \hat{f}(m)1$ for all $f \in H^\infty(D)$. In both cases, $\sigma(T) = \{0, 1\}$.

(b) Finally, let $\{z_n\}$ be an interpolating Blaschke sequence approaching 1, $z_1 = 0$, with m in the w* closure of $\{z_n\}$ and B the corresponding Blaschke product. If $\tau(z) = \frac{1}{2} B(z)$, then it is well known [3] that $(\hat{\tau} \circ L_m)'(0) = \frac{1}{2}(\hat{B} \circ L_m)'(0) \neq 0$. This, then, is an example of a compact endomorphism of $H^\infty(D)$ which is not a composition operator but whose spectrum properly contains $\{0, 1\}$.

References

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This research was supported by EPSRC grant GR/M31132